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Exact solutions in bouncing cosmology

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Abstract

We discuss the effects of a (possibly) negative $(1+z)^6$ type contribution to the Friedmann equation in a spatially flat universe. No definite answer can be given as to the presence and magnitude of a particular mechanism, because any test using the general relation $H(z)$ is able to estimate only the total of all sources of such a term. That is why we describe four possibilities: (1) geometric effects of loop quantum cosmology, (2) braneworld cosmology, (3) metric-affine gravity, and (4) cosmology with spinning fluid. We find the exact solutions for the models with ρ^2 correction in terms of elementary functions, and show all evolutional paths on their phase plane. Instead of the initial singularity, the generic feature is now a bounce.

1. Introduction

Cosmology, mainly due to its modern astronomical observations, seems to have been limited to what is nowadays dubbed the Cosmological Concordance Model (CCM). In the observational cosmology this role is played by the cold dark matter model with the cosmological constant (ΛCDM model). It offers the simplest explanation of the current Universe filled with two components—dark matter and dark energy [1,2]. The former is in the form of non-relativistic dust matter contributing one third of the total energy density. The latter has negative pressure, violating the strong energy condition, and constitutes the remaining two thirds.

Interpretations of observational data within this model lead to the conclusion that we live in an accelerating universe which is almost spatially flat and low density [3–5]. Although the introduction of Λ could be justified on purely geometrical ground, many authors seek a physical source of a suitably behaving matter, mimicking such a geometrical effect. In literature there are many cosmological models with dark energy which take part in the contest for the best description of the accelerating Universe [6,7].

On the other hand, the very early universe, when classical physics is no longer sufficient, is not so well understood. It is believed that "initial conditions" for the classical evolution could be obtained from different, fundamental physics which dominates when the energy density is high enough, and which turn negligible as the Universe expands. In other words, new physics is welcome in the process of cosmological model building [8,9]. In particular, one could hope for avoiding the initial singularity in a new model—for example with loop quantum gravity where a bounce is generic [23].

What we consider here are the effects of theories which predict a ρ^2-type modification to the Friedmann equation, which fits the above description. Although such reduction is a big simplification, we feel it is important to stress that such problems are explicitly solvable, which is often not checked before applying numerical studies in current works; and that it is also possible to find and classify all types of evolutional scenarios which help to gain insight into more complicated models.

Below, we outline some recently significant possibilities of non-classical physics, and proceed to solve the resulting Friedmann equation in the subsequent sections.

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1.1. Braneworlds

The basic idea is that our observational universe is some four-dimensional surface (called a brane) embedded in a more dimensional bulk spacetime, in which the gravitational field, but not the others, can freely propagate. The universe is self accelerating due to an additional term appearing in the Friedmann equation when constrained to the brane (for a pedagogical introduction into extra dimensions cosmology see e.g. [10–12]).

Let us consider a higher-dimensional cosmology in the Randall–Sundrum framework [13,14]. Then the Friedmann equation assumes the following form [15,16]

$$H^2 = \frac{\Lambda_4}{3} + \frac{8\pi}{3M_p^2}\rho + \epsilon\left(\frac{4\pi}{3M_5^3}\right)^2\rho^2 + \frac{c}{a^4}, \tag{1}$$

where Λ_4 is the four-dimensional cosmological constant, $\epsilon = \pm 1$, and c is an integration constant whose magnitude as well as sign depend on the initial conditions. The fifth term (called dark radiation if $c < 0$) we put equal to 0 because the third term is of the main interest in the present discussion. This ρ^2 contribution arises from the imposition of a junction condition for the scale factor of the brane. Note that the ρ^2 term would decay as rapidly as a^{-6} in a matter dominated universe, thus modifying the early evolution and not being significant later on.

Both negative and positive ϵ are possible because ϵ corresponds to the metric signature of the extra dimension [15]. However, the sign of ϵ is crucial because models with timelike extra dimension can avoid initial singularity by the so-called "bounce" [17]. Of course, the sign of ϵ remains an open question—in [18], the authors discuss some consequences of the choice $\epsilon = -1$. Then, the presence of the ρ^2 term on the right-hand side of (1) leads a contracting universe to a bounce instead of a big bang type of curvature singularity ($\rho \to \infty$, $R_{abcd}R^{abcd} \to \infty$). If that happens, we have a constraint on the value of brane tension $|\sigma| \gtrsim (1\ \text{MeV})^4$ because the bounce takes place at densities greater than during nucleosynthesis.

It should also be added that string theory does not provide any reasons for an additional timelike dimension. It is, nevertheless, a mathematically viable possibility, that some authors choose to investigate.

Note that such modification was also recently investigated in the context of two-brane model [19], and such equations arise in classically constrained gravity of Gabadadze and Shang [20,21].

1.2. Loop quantum universes

The standard Big Bang cosmology presents us with the problem of initial conditions for the Universe. What we can observe from "the inside" cannot answer the question why particular conditions were chosen, as we are simply looking at the dynamics. Unfortunately we cannot look from "the outside" and risk any kind of statistical or other external explanation. Many authors shift the problem to the Planckian epoch, during which the quantum effects are important. A contraction phase preceding the present expansion, as predicted by some of these theories, seems to be one of the more attractive scenarios. It is very interesting that the geometric effects in Loop Quantum Cosmology predict a $-\rho^2$ modification to the Friedmann equation [22,23]. This modification is relevant in the very early universe when its matter energy density becomes comparable with the Planck density. The effective Friedmann equation becomes

$$H^2 = \frac{8\pi G\rho}{3}\left(1 - \frac{\rho}{\rho_{\text{crit}}}\right) + \frac{\Lambda}{3}, \tag{2}$$

where $\rho_{\text{crit}} = \sqrt{3}/(16\pi^2\gamma^2)\rho_{\text{pl}}$ [24], and one can distinguish classical and quantum bounces depending on the relative magnitude of ρ and ρ_{crit}.

This is an application of the LQG methods, where the spacetime is discrete on the quantum level, considerably affecting the large scale [25–27] with nonperturbative corrections. The ρ^2 modification is shared with the braneworld model, and some comparison work can be found in [28–30], see also [31].

1.3. Non-Riemannian cosmologies

This misleadingly dubbed concept does not in fact involve the abandonment of Riemannian manifold structure, but the name has been widely used so far and we also adopt it here.

The first non-Riemannian cosmological models were based on the Einstein–Cartan theory, which is a modification of general relativity by adding the torsion of the space–time [32]. In such models, the effects of spin and torsion are manifested by the presence of an additional $-\rho^2 \propto -a^{-6}$ contribution in the Friedmann equation. The main motivation for studying such models was the problem of the initial singularity which could be avoided due to the spin effects.

Among various extensions of general relativity, the so-called metric-affine gravity (MAG) is recently of interest. In contrast to the Riemann–Cartan theory, the connection is no longer metric which implies that the covariant derivative of the metric does not vanish. For a review see e.g. [8,33].

This time, the Friedmann equation reads

$$H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} + \nu\frac{\psi^2}{a^6}, \tag{3}$$

where the new constant ν can be of both signs, and ψ is an integration constant. Therefore, the non-Riemannian quantities of this model (torsion and non-metricity) will modify very early stages of evolution and are negligible at later times.

We find different possible interpretations of the presence of the ρ^2 contribution in the Friedmann equation. However, we must remember that cosmography which is based on the behaviour of the null geodesics maps the geometry and kinematics of the universe in terms of $H(z)$, without any reference to the source of each contribution. This happens because it measures only the average properties of the matter density, and eventually only the overall term $(-)(1+z)^6$ is significant [34].

Interestingly, similar a^{-6} modifications have also been obtained in universes with varying constants [35]. It was shown that in the generic case, alpha varying models lead to a bouncing universe.

2. Dynamics of the model

We take the Friedmann equation in the following form

$$\frac{a'(t)^2}{a(t)^2} = \frac{8\pi G}{3}\rho(t) + \nu\rho(t)^2 + \frac{\Lambda}{3}, \tag{4}$$

with a general ρ^2 term, and a new, real constant ν. As mentioned before, spatially flat space is assumed, although there exist curved models in which the dynamics is also solvable by means of elementary functions [36,37]. We define a scale factor x, so that it is equal to unity at time t_0 (say, today), and accordingly, the scaling law for the (dust) matter density

$$a(t) = a_0 x(t), \qquad \rho(t) = \rho_{m0} x(t)^{-3}. \tag{5}$$

The main equation is now

$$\frac{x'(t)^2}{x(t)^2} = \frac{\Lambda}{3} + \frac{8\pi G\rho_{m0}}{3}x(t)^{-3} + \nu\rho_{m0}^2 x(t)^{-6}. \tag{6}$$

Next, we introduce the density parameters Ω

$$\rho_{m0} = \frac{3H_0^2}{8\pi G}\Omega_{m0}, \qquad \nu\rho_{m0}^2 = H_0^2\Omega_q, \qquad \Lambda = 3H_0^2\Omega_\Lambda, \tag{7}$$

which requires a change of the time variable

$$x(t) = y(H_0 t) = y(s), \tag{8}$$

so that the equation is simplified to a well-known form

$$\frac{y'(s)^2}{y(s)^2} = \Omega_\Lambda + \Omega_{m,0}y(s)^{-3} + \Omega_q y(s)^{-6}, \tag{9}$$

with the condition binding the present values of densities

$$1 = \Omega_\Lambda + \Omega_{m,0} + \Omega_q. \tag{10}$$

Note, that in our notation these Ω's are all constants. Finally, introducing a new dependent variable u

$$y(s) = \sqrt[3]{u(s)} \tag{11}$$

the right-hand side becomes a polynomial

$$\frac{1}{9}u'(s)^2 = \Omega_\Lambda u(s)^2 + \Omega_{m,0}u(s) + \Omega_q, \tag{12}$$

which can immediately be solved by the use of trigonometric functions only as

$$u(s) = \frac{-2\Omega_{m0}\sqrt{\Omega_\Lambda} + (\Omega_\Lambda + \Omega_{m,0}^2 - 4\Omega_\Lambda\Omega_q)\cosh[3\sqrt{\Omega_\Lambda}(\alpha \pm s)] + (\Omega_\Lambda - \Omega_{m,0}^2 + 4\Omega_\Lambda\Omega_q)\sinh[3\sqrt{\Omega_\Lambda}(\alpha \pm s)]}{4\Omega_\Lambda^{9/2}}, \tag{13}$$

where α is the integration constant. Obviously such a general solution is not very clear, the more so because α is in general a complex number. That is why we simplify the main equation a bit more with another set of substitutions (provided that $\Lambda \neq 0$)

$$u(s) = w\big(3\sqrt{|\Omega_\Lambda|}s\big) = w(\tau), \qquad \Omega_q = q|\Omega_\Lambda|, \qquad \Omega_{m0} = \mu|\Omega_\Lambda|, \tag{14}$$

and obtain

$$w'(\tau)^2 = \operatorname{sign}(\Lambda) w(\tau)^2 + \mu w(\tau) + q, \tag{15}$$

or, if $\Lambda = 0$,

$$w'(\tau)^2 = \Omega_{m,0} w(\tau) + 1 - \Omega_{m,0}, \tag{16}$$

upon using (10). Depending on the sign of Λ, we have three main groups of solutions, which we proceed to describe in more detail.

3. $\Lambda > 0$

The curves described by Eq. (15) in the plane (w, w') are either two hyperbolae or straight lines, as shown in Fig. 1. If $\Delta = \mu^2 - 4q$ is negative, we have the hyperbolae situated above and below $e_0 = -\mu/2$—they are really the same solution, only with the sign of time interchanged. The main equation always admits such a symmetry, but as we want a universe which is expanding at this moment, we choose the solution with $w'(0) > 0$. It is given by

$$w(\tau) = \frac{1}{2}\bigl[-\mu + (2+\mu)\cosh(\tau) + 2\sqrt{1+q+\mu}\,\sinh(\tau)\bigr], \tag{17}$$

and shown in Fig. 2.

When $\Delta > 0$, we have the other two hyperbolae, but as $\Omega_{m,0}$ and Λ are both positive, e_0 is negative, and so only the right branch can intersect with the $w = 1$ line. If $e_2 = (\mu + \sqrt{\Delta})/2 > 0$, a bounce without singularity appears. The formula is the same as in (17), and is shown in Fig. 3.

This is in fact the generic setup, as Λ is usually assumed to be positive, and the ρ^2 contribution negative. This means that q is negative, making Δ positive. For this reason we consider bounce the generic feature here.

If $\Delta = 0$, e_0 becomes a double root of the polynomial in the right-hand side of Eq. (15) and accordingly a possible static solution. However, it is negative in our physical setup, so the only possibility are the straight-line solutions. They are different in that they asymptotically evolve from the static solution in the infinite past, as can be seen in Fig. 4. Formula (17) still holds.

4. $\Lambda < 0$

The situation is much simpler now, as shown in Fig. 5—the only possible curve is a circle when $\Delta = \mu^2 - 4q$ is positive. The condition (10) means that the circle must intersect the $w = 1$ line, unless $\Delta = 0$, which corresponds to a stable, static solution. If $e_1 = (-\mu - \sqrt{\Delta})/2$ is positive, the evolution is a singularity free oscillation. Obviously only one formula is needed to describe this case

$$w(\tau) = \frac{1}{2}\bigl[-\mu + (2+\mu)\cos(\tau) + 2\sqrt{-1-q-\mu}\,\sin(\tau)\bigr], \tag{18}$$

and a typical example is shown in Fig. 6.

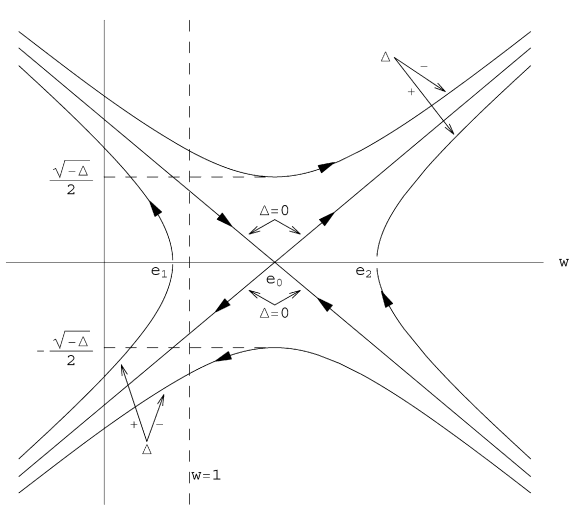

Fig. 1. The possible evolutional paths of the system when $\Lambda > 0$, depending on the sign of $\Delta = \mu^2 - 4q$. Only the paths intersected by the $w = 1$ line are admissible in a given set of the parameters. $e_0 = -\mu/2$, $e_{1,2} = (-\mu \mp \sqrt{\Delta})/2$.

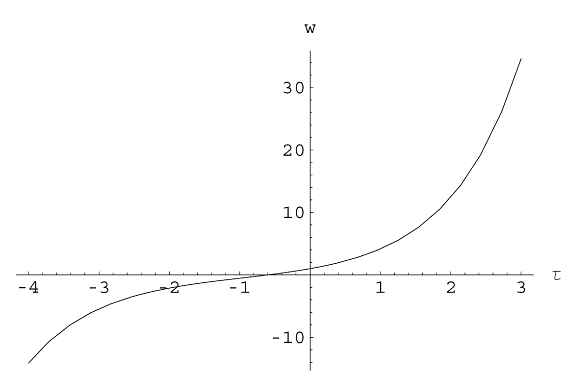

Fig. 2. $\Lambda > 0$ and $\Delta < 0$ solution.

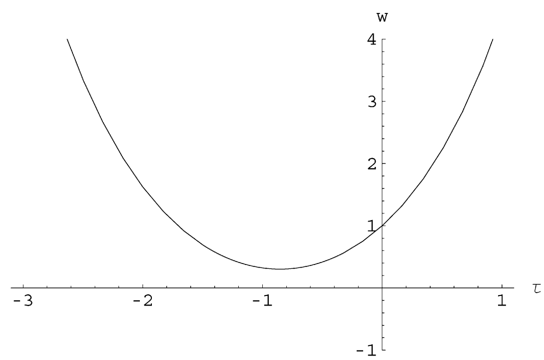

Fig. 3. $\Lambda > 0$ and $\Delta > 0$ solution.

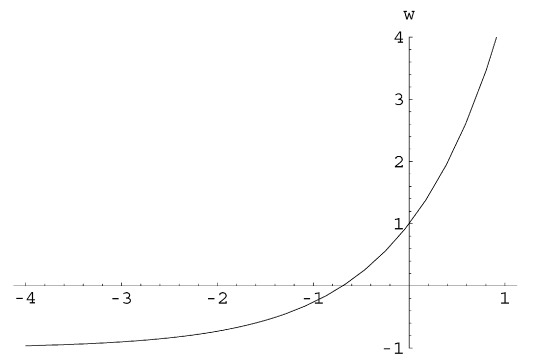

Fig. 4. $\Lambda > 0$ and $\Delta = 0$ solution.

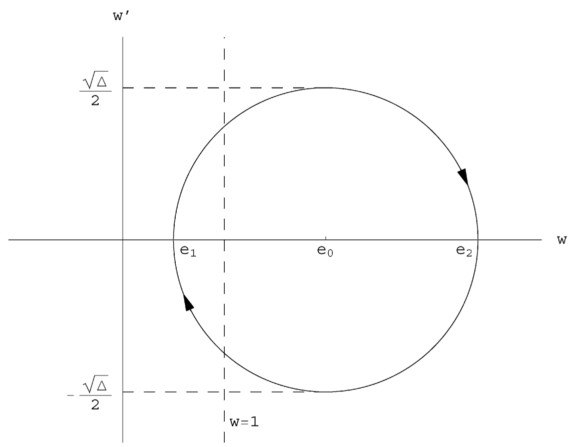

Fig. 5. The possible evolutional paths of the system when $\Lambda < 0$. The circle always intersects the $w = 1$ line, unless $\Delta = 0$, and the solution reduces to the static one $w = e_0 = -\mu/2$, $e_{1,2} = (-\mu \mp \sqrt{\Delta})/2$.

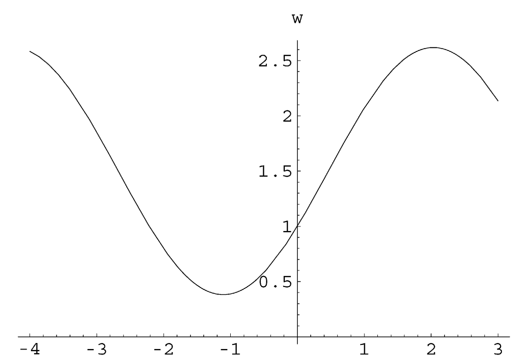

Fig. 6. $\Lambda < 0$ and $\Delta > 0$ solution.

5. $\Lambda = 0$

Finally, the simplest case with zero cosmological constant, where going back to Eq. (16). This time the solution is even simpler—polynomial in time

$$u(s) = 1 + 3s + \frac{9}{4}\Omega_{m,0}s^2, \tag{19}$$

where, as before, we determined the integration constants by assuming $u(0) = 1$ and $u'(0) > 0$. This is also a bounce, with a minimum size of

$$u_{\min} = 1 - \frac{1}{\Omega_{m,0}}, \tag{20}$$

which is only positive if $\Omega_{m,0} > 1$, or equivalently $\Omega_q < 0$, corresponding to a negative ρ^2 term in the Friedmann equation. Qualitatively the behaviour here is the same as that in Fig. 3, only the expansion is not exponentially fast.

6. Conclusions

We have studied the FRW models with an additional $(1+z)^6$ term in the Friedmann equation. Astronomical measurements based on the relation $H(z)$ and general dynamics of such models ignore the number and the particular type of contributions—as long as they scale the same way. Among these, we described: brane theory, non-Riemannian modifications and loop quantum cosmology. Their combined effect, as observed through the Hubble's relation, could in theory be null, due to the indefiniteness of sign. Which is not to say that they would be indistinguishable (or null) when other observations come into account.

Assuming the non-zero term a^{-6}, it is still possible to find exact solutions, and classify all possible evolutional paths. It was shown that exact solutions can be expressed in terms of elementary functions and depending on the sign of Λ we have three main families (with the exact formulas provided in the respective sections).

For $\Lambda = 0$ (Section 5) we obtain a simple algebraic solution which is a bounce, and for a negative ρ^2 contribution the minimal value of the scale factor is positive. Qualitatively it is the same as the one presented in Fig. 3.

When $\Lambda < 0$ (Section 4), we have a periodic behaviour (Fig. 6) of the scale factor, but as above, the minima are not necessarily positive.

Finally, taking $\Lambda > 0$ (Section 3), we obtain a universe which expand exponentially, but which could have followed three different paths in the past. It could have bounced (Fig. 3), it could have been expanding all the time from minus infinity (Fig. 2) or asymptotically from the static solution (Fig. 4). Of course, the last two scenarios, would have to be considered as singular and expanding from the $a = 0$ singularity.

The generic feature seems to be the replacement of the initial singularity by a bounce. However, the effects might influence also the late evolution if one considers a phantom fluid, and are a possibility of avoiding the so-called Big Rip singularity [38].

As mentioned before, similar exact solutions of the Friedmann equation were found for brane scenarios by Setare [36,37]. Such solutions are among the simplest possible, where the left-hand side of the equation is a polynomial of degree 2, or can be reduced to such a polynomial. Usually this can be done by either introducing a new "conformal" time, or choosing a new scale factor which some power of a. For polynomials of greater degrees such reduction is, in general, impossible and elliptic functions have to be used [39].

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